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Generalized Plane Solution for Monoclinic Piezoelectric Laminates

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An efficient method has been developed for analyzing the three-dimensional electromechanical deformations of a monoclinic piezoelectric laminate subjected to surface tractions and electric displacements on the top and bottom surfaces. The procedure combines the transfer matrix method with the asymptotic expansion technique. The electromechanical coupled formulation is reduced to a hierarchy of problems referred to as a reference plane. A closed-form solution is obtained for the generalized plane deformations of a piezoelectric laminate with the top and bottom surfaces subjected to uniform electric displacements and mechanical normal tractions and edges at the midplane rigidly clamped and grounded. The closed-form solution is obtained because the asymptotic expansion terminates after a few terms. For some loadings the in-plane electric fields can be much larger than the out-of-plane electric fields.

I. Introduction

A SMART structural system is a composite structure with surface-mounted or embedded active materials. Beams, plates, and shells are the structural elements most frequently used in aeronautics and aerospace applications. The integrated active structures contain piezoelectric patches and layers that act as sensors and actuators. References 1–4 provide more details of the immense technological potentials and implications of the smart materials and structures.

A piezoelectrically induced strain as a source of actuation may be interpreted as an eigenstrain[2,5–10] entering the constitutive relations. These approaches estimate eigenstrain from the applied voltage and provide no information on the errors involved. Moreover, in certain circumstances[11–13] the in-plane electric field components are more significant than the transverse component, thereby making their omission unjustifiable.

Transfer matrix approaches[14–19] have been developed to study the electromechanical coupling characteristics of laminated piezoelectric plates. However, the basic equations must be reducible to a system of ordinary differential equations in which the thickness coordinate is the only spatial variable. This restricts applications of the method to a small class of problems. An asymptotic expansion method for three-dimensional elastic analysis[20–22] has been successfully developed. An important result is that solutions of the three-dimensional elasticity equations can be generated from solutions of the classical two-dimensional plate equations. In piezoelectricity an asymptotic theory of leading-order approximation for thin single-layer homogeneous piezoelectric plates[23,24] has also been proposed. By extending the elasticity work,[20] a three-dimensional solution of electroelastic elliptic symmetric laminates[12] has been presented. However, all of these electroelasticity approaches appear to be applicable to plates symmetric about the midplane.

Here, the three-dimensional asymptotic approach is further developed for monoclinic piezoelectric laminates, which are not necessarily symmetric about the midplane. General formulations are derived to any successive approximations and used for a generalized plane problem. Neglecting the effects of the boundary layer near the clamped edges, a closed-form solution is obtained at interior points of a piezoelectric laminate subjected to uniform normal tractions and electric displacements on the top and bottom surfaces. This solution essentially generalizes the elasticity solution.[21]

II. Formulation of the Problem

We use rectangular Cartesian coordinates x_i, $(i = 1, 2, 3)$ to describe the infinitesimal electromechanical deformations of a plate of uniform thickness h and take the midplane of the plate to coincide with the plane $x_3 = 0$. The plate is made of a monoclinic piezoelectric material. In the absence of body forces and electric charge density, equations governing the quasistatic electromechanical deformations of the body are[25–27]

$$\tau_{ij,j} = 0, \qquad D_{i,i} = 0 \tag{1}$$

where τ_{ij} is the stress tensor and D_i the electric displacement. Throughout this paper a comma followed by an index i denotes partial differentiation with respect to x_i, a repeated index implies summation over the range of the index, and Latin indices range from 1 to 3 and Greek indices from 1 to 2. The dependence of functions and operators on x_i is not explicitly shown unless necessary. The infinitesimal strain tensor S_{kl} and the electric field E_k are related to the mechanical displacements u_k and the electric potential φ through the gradient relations

$$S_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}), \qquad E_k = -\varphi_{,k} \tag{2}$$

The constitutive relations for a monoclinic piezoelectric material can be written as

$$\tau_{\alpha\beta} = c_{\alpha\beta\omega\rho} S_{\omega\rho} + c_{\alpha\beta 33} S_{33} - e_{3\alpha\beta} E_3$$

$$\tau_{\alpha 3} = 2c_{\alpha 3\omega 3} S_{\omega 3} - e_{\omega\alpha 3} E_\omega$$

$$\tau_{33} = c_{33\omega\rho} S_{\omega\rho} + c_{3333} S_{33} - e_{333} E_3, \qquad D_\alpha = 2e_{\alpha\omega 3} S_{\omega 3} + \varepsilon_{\alpha\omega} E_\omega$$

$$D_3 = e_{3\omega\rho} S_{\omega\rho} + e_{333} S_{33} + \varepsilon_{33} E_3 \tag{3}$$

Here, we have assumed that the material properties are symmetric with respect to the x_1–x_2 plane, i.e., the diad axis is the x_3 direction. Accordingly, the numbers of nonzero independent elastic moduli c_{ijkl}, piezoelectric moduli e_{kij}, and dielectric moduli ε_{ik} are respectively 13, 8, and 4 for the monoclinic material. These material moduli exhibit the following symmetries:

$$c_{ijkl} = c_{jikl} = c_{klij}, \qquad e_{kij} = e_{kji}, \qquad \varepsilon_{ki} = \varepsilon_{ik} \tag{4}$$

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For a laminated plate with laminae made of different homogeneous monoclinic piezoelectric materials, the material moduli are piecewise constant functions of x_3.

By eliminating the possibly discontinuous in-plane stresses $\tau_{\alpha\beta}$ and in-plane electric displacements D_ρ from Eqs. (1) and (3), and expressing $S_{\omega 3}$ in Eq. (3) in terms of $\tau_{\alpha 3}$ and φ, a state-space equation for the piezoelectric plate is formulated as

$$\partial_3 \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{bmatrix} \tag{5}$$

where $\partial_3 \equiv \partial/\partial x_3$ and the state-space variables are chosen as

$$\boldsymbol{F} = \begin{bmatrix} u_1 \\ u_2 \\ \tau_{33} \\ D_3 \end{bmatrix}, \qquad \boldsymbol{G} = \begin{bmatrix} \tau_{13} \\ \tau_{23} \\ u_3 \\ \varphi \end{bmatrix} \tag{6}$$

The 4×4 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in the state-space equation (5) are differential operator matrices:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{J}_\beta \partial_\beta \\ -\boldsymbol{J}_\beta^T \partial_\beta & \boldsymbol{K}_{\beta\rho}\partial_\beta\partial_\rho \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} -\boldsymbol{L}_{\beta\rho}\partial_\beta\partial_\rho & -\boldsymbol{M}_\beta \partial_\beta \\ -\boldsymbol{M}_\beta^T \partial_\beta & \boldsymbol{N} \end{bmatrix} \tag{7}$$

where both $\boldsymbol{A}$ and $\boldsymbol{B}$ have been partitioned into four 2×2 operator submatrices and

$$\boldsymbol{I} \equiv (I^{\omega\alpha}) \equiv \begin{bmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{bmatrix}, \qquad \boldsymbol{J}_\beta \equiv (J_\beta^{\omega\alpha}) \equiv \begin{bmatrix} J_\beta^{11} & J_\beta^{12} \\ J_\beta^{21} & J_\beta^{22} \end{bmatrix}$$

$$\boldsymbol{K}_{\beta\rho} \equiv (K_{\beta\rho}^{\omega\alpha}) \equiv \begin{bmatrix} K_{\beta\rho}^{11} & K_{\beta\rho}^{12} \\ K_{\beta\rho}^{21} & K_{\beta\rho}^{22} \end{bmatrix}, \qquad \boldsymbol{L}_{\beta\rho} \equiv (L_{\beta\rho}^{\alpha\omega}) \equiv \begin{bmatrix} L_{\beta\rho}^{11} & L_{\beta\rho}^{12} \\ L_{\beta\rho}^{21} & L_{\beta\rho}^{22} \end{bmatrix}$$

$$\boldsymbol{M}_\beta \equiv (M_\beta^{\alpha\omega}) \equiv \begin{bmatrix} M_\beta^{11} & M_\beta^{12} \\ M_\beta^{21} & M_\beta^{22} \end{bmatrix}, \qquad \boldsymbol{N} \equiv (N^{\alpha\omega}) \equiv \begin{bmatrix} N^{11} & N^{12} \\ N^{21} & N^{22} \end{bmatrix} \tag{8}$$

For a laminated plate $\boldsymbol{A}$ and $\boldsymbol{B}$ are piecewise constant functions of x_3; the submatrices are only related to the material moduli, and their elements can be scalars, vectors, and tensors. We use superscripts to denote the location, i.e., row and column, of matrix elements in Eq. (8), whereas the Greek subscripts are for the usual tensor notation. These elements are expressed in terms of the material moduli as

$$[J_\beta^{\omega 1} \quad J_\beta^{\omega 2}] = [\delta_{\omega\beta} \quad I^{\omega\alpha} e_{\beta\alpha 3}]$$

$$K_{\beta\rho}^{11} = K_{\beta\rho}^{12} = K_{\beta\rho}^{21} = 0, \qquad K_{\beta\rho}^{22} = I^{\omega\alpha} e_{\beta\alpha 3} e_{\rho\omega 3} + \varepsilon_{\beta\rho}$$

$$L_{\beta\rho}^{\alpha\omega} = c_{\alpha\beta\omega\rho} - [c_{\alpha\beta 33} \quad e_{3\alpha\beta}]\boldsymbol{N} \begin{bmatrix} c_{33\omega\rho} \\ e_{3\omega\rho} \end{bmatrix}$$

$$[M_\beta^{\alpha 1} \quad M_\beta^{\alpha 2}] = [c_{\alpha\beta 33} \quad e_{3\alpha\beta}]\boldsymbol{N} \tag{9}$$

where $\delta_{\omega\beta}$ is the Kronecker delta and

$$\boldsymbol{I}^{-1} = \begin{bmatrix} c_{1313} & c_{1323} \\ c_{1323} & c_{2323} \end{bmatrix}, \qquad \boldsymbol{N}^{-1} = \begin{bmatrix} c_{3333} & e_{333} \\ e_{333} & -\varepsilon_{33} \end{bmatrix} \tag{10}$$

The in-plane stresses and in-plane electric displacements, which may be discontinuous in x_3, are then given by

$$\tau_{\alpha\beta} = L_{\beta\rho}^{\alpha\omega}\partial_\rho u_\omega + M_\beta^{\alpha 1}\tau_{33} + M_\beta^{\alpha 2} D_3$$

$$D_\rho = J_\rho^{\alpha 2}\tau_{\alpha 3} - K_{\beta\rho}^{22}\partial_\beta\varphi \tag{11}$$

The state-space equation (5) for piezoelectricity is structurally the same as that for pure elasticity except for the contribution from the two additional variables because of the electric field. The state-space equation for an elastic plate can be recovered from Eq. (5) by setting $e_{kij} = 0$.

We introduce a dimensionless parameter $\varepsilon = h/2a$, where a is a typical in-plane length of the plate and set $z = x_3/\varepsilon$. Thus z varies from $-a$ to a as x_3 goes from $-h/2$ to $h/2$. The state-space equation (5) can now be written as

$$\partial_z \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{bmatrix} = \varepsilon \begin{bmatrix} 0 & \boldsymbol{A} \\ \boldsymbol{B} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{bmatrix} \tag{12}$$

and has the formal solution[28]

$$\begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{bmatrix} = \boldsymbol{P} \begin{bmatrix} \boldsymbol{F}(z^*) \\ \boldsymbol{G}(z^*) \end{bmatrix} \tag{13}$$

where z^* takes a specific value of z and hence the initial functions $\boldsymbol{F}(z^*)$ and $\boldsymbol{G}(z^*)$ are functions of x_α only. Their components are the unknowns to be determined from specified surface and edge conditions. The transfer matrix $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \sum_{n=0}^{\infty} \varepsilon^{2n} \begin{bmatrix} \boldsymbol{a}^{(2n)} & \varepsilon \boldsymbol{a}^{(2n+1)} \\ \varepsilon \boldsymbol{b}^{(2n+1)} & \boldsymbol{b}^{(2n)} \end{bmatrix} \tag{14}$$

where

$$\boldsymbol{a}^{(n+1)} = \int_{z^*}^{z} \boldsymbol{A}\boldsymbol{b}^{(n)}\,\mathrm{d}z, \qquad \boldsymbol{b}^{(n+1)} = \int_{z^*}^{z} \boldsymbol{B}\boldsymbol{a}^{(n)}\,\mathrm{d}z \qquad (n \geq 0) \tag{15}$$

$$\boldsymbol{a}^{(0)} = \boldsymbol{b}^{(0)} = \boldsymbol{i} \tag{16}$$

is a 4×4 identity matrix. Similar to $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{a}^{(n)}$ and $\boldsymbol{b}^{(n)}$ are the operator matrices that include differential operators only with respect to x_α, whereas their variation with z appears only in the material moduli.

We expand the state-space variables $\boldsymbol{F}$ and $\boldsymbol{G}$ in terms of the small parameter ε as

$$\begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{bmatrix} = \sum_{n=0}^{\infty} \varepsilon^{2n} \begin{bmatrix} \varepsilon \boldsymbol{f}^{(n)} \\ \boldsymbol{g}^{(n)} \end{bmatrix} \tag{17}$$

Substitution from Eqs. (14) and (17) into Eq. (13) yields

$$\boldsymbol{f}^{(n)} = \sum_{k=0}^{n} [\boldsymbol{a}^{(2k)}\boldsymbol{f}^{(n-k)}(z^*) + \boldsymbol{a}^{(2k+1)}\boldsymbol{g}^{(n-k)}(z^*)] \qquad (n \geq 0)$$

$$\boldsymbol{g}^{(0)} = \boldsymbol{g}^{(0)}(z^*)$$

$$\boldsymbol{g}^{(n)} = \sum_{k=0}^{n-1} \boldsymbol{b}^{(2k+1)}\boldsymbol{f}^{(n-1-k)}(z^*) + \sum_{k=0}^{n} \boldsymbol{b}^{(2k)}\boldsymbol{g}^{(n-k)}(z^*) \qquad (n \geq 1) \tag{18}$$

Equations (18) furnish general expressions for the nth-order expansion coefficients of the field functions in terms of the expansion coefficients of the initial functions from the zeroth order to the nth order. Hence the three-dimensional problem reduces to a series of two-dimensional problems defined on the reference plane. As for most plate theories, we designate the reference plane $z = z^*$ to be the midplane, i.e., $z^* = 0$.

III. Governing Equations of Initial Functions

We assume that the top and bottom surfaces of a piezoelectric plate are subjected to given shear tractions $q_\alpha^\pm$, normal tractions $-q_3^\pm$, and normal electric displacements $D_3^\pm$, which are scaled as

$$\tau_{\alpha 3}(\pm a) = \varepsilon^2 q_\alpha^\pm, \qquad \tau_{33}(\pm a) = -\varepsilon^3 q_3^\pm, \qquad D_3(\pm a) = \varepsilon D_3^\pm \tag{19}$$

Thus functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in Eq. (17) are given by

$$g_\alpha^{(0)} = \tau_{\alpha 3}^{(0)} = 0, \qquad f_3^{(0)} = \tau_{33}^{(0)} = 0 \tag{20}$$

$$g_\alpha^{(n+1)}(\pm a) = \tau_{\alpha 3}^{(n+1)}(\pm a) = q_\alpha^\pm \delta_{n0}$$

$$f_3^{(n+1)}(\pm a) = \tau_{33}^{(n+1)}(\pm a) = -q_3^\pm \delta_{n0}$$

$$f_4^{(n)}(\pm a) = D_3^{(n)}(\pm a) = D_3^\pm \delta_{n0} \qquad (n \geq 0) \tag{21}$$

Using Eqs. (16) and (18) and $a_{33}^{(1)} = a_{34}^{(1)} = a_{43}^{(1)} = 0$, Eqs. (21) can be rewritten as

$$\tau_{\alpha 3}^{(n+1)}(0) + b_{\alpha\omega}^{(1)}(\pm a)U_\omega^{(n)} + b_{\alpha 3}^{(1)}(\pm a)\tau_{33}^{(n)}(0) + b_{\alpha 4}^{(1)}(\pm a)D_3^{(n)}(0)$$
$$+ b_{\alpha\omega}^{(2)}(\pm a)\tau_{\alpha 3}^{(n)}(0) + b_{\alpha 3}^{(2)}(\pm a)U_3^{(n)} + b_{\alpha 4}^{(2)}(\pm a)\Phi^{(n)}$$
$$+ \sum_{k=1}^{n}\left[b_{\alpha L}^{(2k+1)}(\pm a)f_L^{(n-k)}(0) + b_{\alpha L}^{(2k+2)}(\pm a)g_L^{(n-k)}(0)\right]$$
$$= q_\alpha^\pm \delta_{n0}$$

$$\tau_{33}^{(n+1)}(0) + a_{3\alpha}^{(1)}(\pm a)\tau_{\alpha 3}^{(n+1)}(0) + a_{3\omega}^{(2)}(\pm a)U_\omega^{(n)} + a_{33}^{(2)}(\pm a)\tau_{33}^{(n)}(0)$$
$$+ a_{34}^{(2)}(\pm a)D_3^{(n)}(0) + a_{3\alpha}^{(3)}(\pm a)\tau_{\alpha 3}^{(n)}(0) + a_{33}^{(3)}(\pm a)U_3^{(n)}$$
$$+ a_{34}^{(3)}(\pm a)\Phi^{(n)} + \sum_{k=1}^{n}\left[a_{3L}^{(2k+2)}(\pm a)f_L^{(n-k)}(0)\right.$$
$$\left.+ a_{3L}^{(2k+3)}(\pm a)g_L^{(n-k)}(0)\right] = -q_3^\pm\delta_{n0}$$

$$D_3^{(n)}(0) + a_{4\alpha}^{(1)}(\pm a)\tau_{\alpha 3}^{(n)}(0) + a_{44}^{(1)}(\pm a)\Phi^{(n)}$$
$$+ \sum_{k=1}^{n}\left[a_{4L}^{(2k)}(\pm a)f_L^{(n-k)}(0) + a_{4L}^{(2k+1)}(\pm a)g_L^{(n-k)}(0)\right]$$
$$= D_3^\pm\delta_{n0} \tag{22}$$

where an uppercase subscript L takes values from 1 to 4 and

$$U_\omega^{(n)} \equiv u_\omega^{(n)}(x_\rho, 0), \qquad U_3^{(n)} \equiv u_3^{(n)}(x_\rho, 0), \qquad \Phi^{(n)} \equiv \varphi^{(n)}(x_\rho, 0) \tag{23}$$

By eliminating four unknowns $\tau_{\alpha 3}^{(n+1)}(0)$, $\tau_{33}^{(n+1)}(0)$, and $D_3^{(n)}(0)$ from Eqs. (22) and using the notation

$$\hat{\boldsymbol{a}}^{(n)} \equiv \boldsymbol{a}^{(n)}(a) - \boldsymbol{a}^{(n)}(-a), \qquad \bar{\boldsymbol{a}}^{(n)} \equiv \boldsymbol{a}^{(n)}(a)$$
$$\hat{\boldsymbol{b}}^{(n)} \equiv \boldsymbol{b}^{(n)}(a) - \boldsymbol{b}^{(n)}(-a), \qquad \bar{\boldsymbol{b}}^{(n)} \equiv \boldsymbol{b}^{(n)}(a) \tag{24}$$

we obtain the following equations:

$$\hat{b}_{\alpha\omega}^{(1)}U_\omega^{(n)} + \hat{b}_{\alpha 3}^{(2)}U_3^{(n)} + \left[\hat{b}_{\alpha 4}^{(2)} - \hat{b}_{\alpha 4}^{(1)}\bar{a}_{44}^{(1)}\right]\Phi^{(n)}$$
$$= (q_\alpha^+ - q_\alpha^-)\delta_{n0} - \hat{b}_{\alpha 4}^{(1)}D_3^+\delta_{n0} - \hat{b}_{\alpha 3}^{(1)}\tau_{33}^{(n)}(0)$$
$$- \left[\hat{b}_{\alpha\omega}^{(2)} - \hat{b}_{\alpha 4}^{(1)}\bar{a}_{4\omega}^{(1)}\right]\tau_{\omega 3}^{(n)}(0) - \sum_{k=1}^{n}\left\{\left[\hat{b}_{\alpha L}^{(2k+1)} - \hat{b}_{\alpha 4}^{(1)}\bar{a}_{4L}^{(2k)}\right]f_L^{(n-k)}(0)\right.$$
$$\left.+ \left[\hat{b}_{\alpha L}^{(2k+2)} - \hat{b}_{\alpha 4}^{(1)}\bar{a}_{4L}^{(2k+1)}\right]g_L^{(n-k)}(0)\right\}$$

$$\left[\hat{a}_{3\omega}^{(2)} - \hat{a}_{3\alpha}^{(1)}\bar{b}_{\alpha\omega}^{(1)}\right]U_\omega^{(n)} + \left[\hat{a}_{33}^{(3)} - \hat{a}_{3\alpha}^{(1)}\bar{b}_{\alpha 3}^{(2)}\right]U_3^{(n)}$$
$$+ \left\{\hat{a}_{34}^{(3)} - \hat{a}_{34}^{(2)}\bar{a}_{44}^{(1)} - \hat{a}_{3\alpha}^{(1)}\left[\bar{b}_{\alpha 4}^{(2)} - \bar{b}_{\alpha 4}^{(1)}\bar{a}_{44}^{(1)}\right]\right\}\Phi^{(n)}$$
$$= -(q_3^+ - q_3^-)\delta_{n0} - \hat{a}_{3\alpha}^{(1)}q_\alpha^+\delta_{n0} - \left[\hat{a}_{34}^{(2)} - \hat{a}_{3\alpha}^{(1)}\bar{b}_{\alpha 4}^{(1)}\right]D_3^+\delta_{n0}$$
$$- \left[\hat{a}_{33}^{(2)} - \hat{a}_{3\alpha}^{(1)}\bar{b}_{\alpha 3}^{(1)}\right]\tau_{33}^{(n)}(0) - \left\{\hat{a}_{3\omega}^{(3)} - \hat{a}_{34}^{(2)}\bar{a}_{4\omega}^{(1)}\right.$$
$$\left.- \hat{a}_{3\alpha}^{(1)}\left[\bar{b}_{\alpha\omega}^{(2)} - \bar{b}_{\alpha 4}^{(1)}\bar{a}_{4\omega}^{(1)}\right]\right\}\tau_{\omega 3}^{(n)}(0) - \sum_{k=1}^{n}\left\{\hat{a}_{3L}^{(2k+2)} - \hat{a}_{3\alpha}^{(1)}\bar{b}_{\alpha L}^{(2k+1)}\right.$$
$$\left.- \left[\hat{a}_{34}^{(2)} - \hat{a}_{3\alpha}^{(1)}\bar{b}_{\alpha 4}^{(1)}\right]\bar{a}_{4L}^{(2k)}\right\}f_L^{(n-k)}(0) - \sum_{k=1}^{n}\left\{\hat{a}_{3L}^{(2k+3)} - \hat{a}_{3\alpha}^{(1)}\bar{b}_{\alpha L}^{(2k+2)}\right.$$
$$\left.- \left[\hat{a}_{34}^{(2)} - \hat{a}_{3\alpha}^{(1)}\bar{b}_{\alpha 4}^{(1)}\right]\bar{a}_{4L}^{(2k+1)}\right\}g_L^{(n-k)}(0)$$

$$\hat{a}_{44}^{(1)}\Phi^{(n)} = (D_3^+ - D_3^-)\delta_{n0} - \hat{a}_{4\alpha}^{(1)}\tau_{\alpha 3}^{(n)}(0)$$
$$- \sum_{k=1}^{n}\left[\hat{a}_{4L}^{(2k)}f_L^{(n-k)}(0) + \hat{a}_{4L}^{(2k+1)}g_L^{(n-k)}(0)\right] \tag{25}$$

for the four unknowns $U_\omega^{(n)}$, $U_3^{(n)}$, and $\Phi^{(n)}$. The right-hand sides of Eqs. (25) involve $\tau_{\alpha 3}^{(n)}(0)$, $\tau_{33}^{(n)}(0)$, $\boldsymbol{f}^{(k)}(0)$, and $\boldsymbol{g}^{(k)}(0)$, $(k = 0, \ldots, n-1)$, which are determined from solutions of order up to $(n-1)$. The $(n+1)$th-order out-of-plane stresses on the midplane, $\tau_{\alpha 3}^{(n+1)}(0)$ and $\tau_{33}^{(n+1)}(0)$, can be solved from Eq. (22).

With the notations

$$Q(\cdots) \equiv \int_0^z (\cdots)\,\mathrm{d}z, \qquad \bar{Q}(\cdots) \equiv \int_0^a (\cdots)\,\mathrm{d}z$$
$$\hat{Q}(\cdots) \equiv \int_{-a}^{a} (\cdots)\,\mathrm{d}z \tag{26}$$

the equations of leading order $(n = 0)$ can be written as

$$\boldsymbol{C}\boldsymbol{X}^{(0)} = \boldsymbol{Y}^{(0)} \tag{27}$$

where the components of the differential operator $\boldsymbol{C}$ are

$$C_{\alpha\omega} = -\hat{Q}L_{\beta\rho}^{\alpha\omega}\partial_\beta\partial_\rho, \qquad C_{\alpha 3} = \hat{Q}zL_{\beta\rho}^{\alpha\omega}\partial_\beta\partial_\omega\partial_\rho$$
$$C_{\alpha 4} = \hat{Q}\left[L_{\beta\rho}^{\alpha\omega}QJ_\lambda^{\omega 2} - M_\lambda^{\alpha 2}(Q - \bar{Q})K_{\beta\rho}^{22}\right]\partial_\beta\partial_\rho\partial_\lambda$$
$$C_{3\omega} = -\hat{Q}zL_{\beta\rho}^{\alpha\omega}\partial_\alpha\partial_\beta\partial_\rho, \qquad C_{33} = \hat{Q}z^2L_{\beta\rho}^{\alpha\omega}\partial_\alpha\partial_\beta\partial_\omega\partial_\rho$$
$$C_{34} = \hat{Q}z\left[L_{\beta\rho}^{\alpha\omega}QJ_\lambda^{\omega 2} - M_\lambda^{\alpha 2}(Q - \bar{Q})K_{\beta\rho}^{22}\right]\partial_\alpha\partial_\beta\partial_\rho\partial_\lambda$$
$$C_{4\omega} = C_{43} = 0, \qquad C_{44} = \hat{Q}K_{\beta\rho}^{22}\partial_\beta\partial_\rho \tag{28}$$

$$\boldsymbol{X}^{(0)} = \left[U_1^{(0)} \quad U_2^{(0)} \quad U_3^{(0)} \quad \Phi^{(0)}\right]^T \tag{29}$$

$$Y_\alpha^{(0)} = q_\alpha^+ - q_\alpha^- + \hat{Q}M_\beta^{\alpha 2}D_{3,\beta}^+$$
$$Y_3^{(0)} = -(q_3^+ - q_3^-) + a(q_\alpha^+ + q_\alpha^-)_{,\alpha} + \hat{Q}zM_\beta^{\alpha 2}D_{3,\alpha\beta}^+$$
$$Y_4^{(0)} = D_3^+ - D_3^- \tag{30}$$

We notice that the electric potential $\Phi^{(0)}$ can first be solved from Eq. $(27)_4$, and the leading-order terms of the midplane displacements can then be solved from the remaining Eqs. $(27)_{1-3}$. If the plate is symmetric about its midplane, then $C_{\alpha 3} = C_{\alpha 4} = C_{3\omega} = 0$, and Eq. (27) can be further simplified to decouple the in-plane displacements $U_\omega^{(0)}$ from $U_3^{(0)}$ and $\Phi^{(0)}$. This special case has been studied.[12]

For an elastic plate the differential operator matrix $\boldsymbol{C}$ given in Eq. (28) reduces to that of the classical plate theory[29,30] for the bending of a thin monoclinic plate or laminate. For a piezoelectric plate Eq. (27) can be viewed as a set of classical equations governing the unknowns (29) under the action of surface tractions and electric displacements. The higher-order equations (25) show that all of the terms on the right-hand side are expressed as derivatives of the lower-order solutions. Accordingly, the preceding procedure may be used to generate an accurate three-dimensional solution provided that the solution of the two-dimensional classical plate equations has been found.

Furthermore, we note that the left-hand sides of Eqs. (25) of different orders contain the same coefficients, which, after a little simplification, turn out precisely to be the classical operators generalized to piezoelectricity. In principle, any numerical technique for solving the classical plate equations can simply be used to solve the three-dimensional problems for laminated piezoelectric plates.

When the unknowns $\boldsymbol{X}^{(0)}$ can be expressed as polynomials in x_α, then the right-hand sides of Eqs. (25) identically vanish for all orders greater than a certain integer. In such cases the series for the initial functions $\boldsymbol{F}(0)$ and $\boldsymbol{G}(0)$ have a finite number of terms, and $\boldsymbol{F}$ and $\boldsymbol{G}$ become exact solutions of the governing equations for all values of ε.

IV. Example

We consider a piezoelectric strip ($|x_1| \le a$, $x_2 \to \infty$) and assume that the solution is independent of x_2. The matrix operators (7) simplify to

$$A = \begin{bmatrix} I & -J_1\partial_1 \\ -J_1^T\partial_1 & K_{11}\partial_1^2 \end{bmatrix}, \qquad B = \begin{bmatrix} -L_{11}\partial_1^2 & -M_1\partial_1 \\ -M_1^T\partial_1 & N \end{bmatrix} \quad (31)$$

and the field equations (27) for the leading order reduce to

$$d_{\alpha\omega}U^{(0)}_{\omega,11} + d_{\alpha 3}U^{(0)}_{3,111} + d_{\alpha 4}\Phi^{(0)}_{,111} = q^+_\alpha - q^-_\alpha + \hat{Q}M_1^{\alpha 2}D^+_{3,1}$$

$$d_{3\omega}U^{(0)}_{\omega,111} + d_{33}U^{(0)}_{3,1111} + d_{34}\Phi^{(0)}_{,1111} = -(q_3^+ - q_3^-) + a(q_1^+ + q_1^-)_{,1} + \hat{Q}zM_1^{12}D^+_{3,11}$$

$$d_{44}\Phi^{(0)}_{,11} = D_3^+ - D_3^- \quad (32)$$

where

$$d_{\alpha\omega} = -\hat{Q}L_{11}^{\alpha\omega}, \qquad d_{\alpha 3} = \hat{Q}zL_{11}^{\alpha 1},$$

$$d_{\alpha 4} = \hat{Q}\left[L_{11}^{\alpha\omega}QJ_1^{\omega 2} - M_1^{\alpha 2}(Q-\bar{Q})K_{11}^{22}\right], \qquad d_{3\omega} = -\hat{Q}zL_{11}^{1\omega}$$

$$d_{33} = \hat{Q}z^2L_{11}^{11}, \qquad d_{34} = \hat{Q}z\left[L_{11}^{1\omega}QJ_1^{\omega 2} - M_1^{12}(Q-\bar{Q})K_{11}^{22}\right]$$

$$d_{44} = \hat{Q}K_{11}^{22} \quad (33)$$

We assume that the transverse surfaces of the piezoelectric strip are subjected to uniform normal tractions $-q_3^\pm$ and uniform normal electric displacements $D_3^\pm$, with vanishing shear tractions $q_\alpha^\pm$. The edges $x_1 = \pm a$ of the strip are rigidly clamped. As is usually assumed in plate theories, these edge conditions are imposed by requiring that

$$u(\pm a, 0) = 0, \qquad \partial_1 u_3(\pm a, 0) = 0, \qquad \varphi(\pm a, 0) = 0 \quad (34)$$

The electric edge condition $(34)_3$ implies that the edges are grounded at the midplane $z = 0$. These are not pointwise conditions and thus exclude the boundary-layer effect. The edge conditions (34) may be written in the form of their expansion coefficients as

$$U^{(n)}_\omega = 0, \qquad U^{(n)}_3 = 0, \qquad \partial_1 U^{(n)}_3 = 0, \qquad \Phi^{(n)} = 0$$

$$(n \ge 0) \quad \text{on} \quad x_1 = \pm a \quad (35)$$

The solution satisfying the preceding edge conditions is

$$U^{(0)}_\omega = k_\omega x_1(x_1^2 - a^2), \qquad U^{(0)}_3 = k_3(x_1^2 - a^2)^2$$

$$\Phi^{(0)} = l(x_1^2 - a^2) \quad (36)$$

where

$$k_\omega = -\tfrac{1}{6}(q_3^+ - q_3^-)\tilde{d}_{\omega 3}, \qquad k_3 = -\tfrac{1}{24}(q_3^+ - q_3^-)\tilde{d}_{33}$$

$$l = (1/2d_{44})(D_3^+ - D_3^-) \quad (37)$$

Here $\tilde{d}$ is the inverse of the submatrix consisting of the first three rows and first three columns of d. The solution of the other four unknowns in Eq. (22) for $n = 0$ can then be obtained as

$$\tau^{(1)}_{\alpha 3}(0) = 6\bar{Q}(k_\omega - 4k_3z\delta_{\omega 1})L_{11}^{\alpha\omega}x_1$$

$$\tau^{(1)}_{33}(0) = s_3 \equiv -q_3^+ + 6\bar{Q}z(k_\omega - 4k_3z\delta_{\omega 1})L_{11}^{1\omega}$$

$$D^{(0)}_3(0) = D \equiv D_3^+ - 2l\bar{Q}K_{11}^{22} \quad (38)$$

Equation $(25)_3$ for $n = 1$ can be rewritten as

$$\hat{a}^{(1)}_{44}\Phi^{(1)} = -\hat{a}^{(1)}_{4\alpha}\tau^{(1)}_{\alpha 3}(0) - \hat{a}^{(2)}_{4\omega}U^{(0)}_\omega - \hat{a}^{(3)}_{43}U^{(0)}_3 \quad (39)$$

whose solution is

$$\Phi^{(1)} = m(x_1^2 - a^2) \quad (40)$$

with

$$m = -(3/d_{44})\hat{Q}\left[J_1^{\alpha 2}(Q-\bar{Q})(k_\omega - 4k_3z\delta_{\omega 1})L_{11}^{\alpha\omega} - K_{11}^{22}Q(k_\omega - 4k_3z\delta_{\omega 1})M_1^{\omega 2}\right] \quad (41)$$

Then Eq. $(22)_3$ for $n = 1$ gives

$$D^{(1)}_3(0) = D' \equiv 6\left(\frac{\bar{Q}K_{11}^{22}}{d_{44}}\hat{Q} - \bar{Q}\right)\left[J_1^{\alpha 2}(Q-\bar{Q})(k_\omega - 4k_3z\delta_{\omega 1})L_{11}^{\alpha\omega} - K_{11}^{22}Q(k_\omega - 4k_3z\delta_{\omega 1})M_1^{\omega 2}\right] \quad (42)$$

The same procedure may be continued for higher-orders terms. A detailed examination of Eq. (22) reveals that all higher-order unknowns identically vanish. Consequently, the nonvanishing expansion coefficients of the initial functions are

$$f^{(0)}(0) = \begin{bmatrix} U^{(0)}_1 \\ U^{(0)}_2 \\ 0 \\ D^{(0)}_3(0) \end{bmatrix}, \qquad g^{(0)}(0) = \begin{bmatrix} 0 \\ 0 \\ U^{(0)}_3 \\ \Phi^{(0)} \end{bmatrix}$$

$$f^{(1)}(0) = \begin{bmatrix} 0 \\ 0 \\ \tau^{(1)}_{33}(0) \\ D^{(1)}_3(0) \end{bmatrix}, \qquad g^{(1)}(0) = \begin{bmatrix} \tau^{(1)}_{13}(0) \\ \tau^{(1)}_{23}(0) \\ 0 \\ \Phi^{(1)} \end{bmatrix} \quad (43)$$

The expansion series for the reference displacements terminates at their first term, while the expansion series for the reference electric potential terminates at its second term. Thus, the evaluation of the reference solution is complete, and the full through-thickness solution of all nonvanishing orders is obtained by substituting these reference values into Eq. (18):

$$g^{(0)} = g^{(0)}(0), \qquad f^{(0)} = f^{(0)}(0) + a^{(1)}g^{(0)}(0)$$

$$g^{(1)} = g^{(1)}(0) + b^{(1)}f^{(0)}(0) + b^{(2)}g^{(0)}(0)$$

$$f^{(1)} = f^{(1)}(0) + a^{(1)}g^{(1)}(0) + a^{(2)}f^{(0)}(0) + a^{(3)}g^{(0)}(0)$$

$$g^{(2)} = b^{(1)}f^{(1)}(0) + b^{(2)}g^{(1)}(0) + b^{(3)}f^{(0)}(0) + b^{(4)}g^{(0)}(0) \quad (44)$$

Any of the through-thickness physical quantities can be evaluated from the reference functions by straightforward differentiations with respect to x_1 and integrations with respect to z (or x_3).

Expressions (17) together with Eqs. (43) and (44) give the complete solution throughout the plate as

$$u_\omega = \varepsilon u^{(0)}_\omega + \varepsilon^3 u^{(1)}_\omega = \varepsilon\left[U^{(0)}_\omega + a^{(1)}_{\omega 3}U^{(0)}_3 + a^{(1)}_{\omega 4}\Phi^{(0)}\right] + \varepsilon^3\left\{\left[a^{(2)}_{\omega\alpha} - a^{(1)}_{\omega\nu}\bar{b}^{(1)}_{\nu\alpha}\right]U^{(0)}_\alpha + \left[a^{(3)}_{\omega 3} - a^{(1)}_{\omega\alpha}\bar{b}^{(2)}_{\alpha 3}\right]U^{(0)}_3 + a^{(1)}_{\omega 4}\Phi^{(1)}\right\}$$

$$\tau_{33} = \varepsilon^3\tau^{(1)}_{33} = \varepsilon^3\left\{\tau^{(1)}_{33}(0) + \left[a^{(2)}_{3\alpha} - a^{(1)}_{3\nu}\bar{b}^{(1)}_{\nu\alpha}\right]U^{(0)}_\alpha + \left[a^{(3)}_{33} - a^{(1)}_{3\alpha}\bar{b}^{(2)}_{\alpha 3}\right]U^{(0)}_3\right\}$$

$$D_3 = \varepsilon D^{(0)}_3 + \varepsilon^3 D^{(1)}_3 = \varepsilon\left[D^{(0)}_3(0) + a^{(1)}_{44}\Phi^{(0)}\right] + \varepsilon^3\left\{D^{(1)}_3(0) + \left[a^{(2)}_{4\alpha} - a^{(1)}_{4\nu}\bar{b}^{(1)}_{\nu\alpha}\right]U^{(0)}_\alpha + \left[a^{(3)}_{43} - a^{(1)}_{4\alpha}\bar{b}^{(2)}_{\alpha 3}\right]U^{(0)}_3 + a^{(1)}_{44}\Phi^{(1)}\right\}$$

$$\tau_{\alpha 3} = \varepsilon^2\tau^{(1)}_{\alpha 3} = \varepsilon^2\left\{\left[b^{(1)}_{\alpha\omega} - \bar{b}^{(1)}_{\alpha\omega}\right]U^{(0)}_\omega + \left[b^{(2)}_{\alpha 3} - \bar{b}^{(2)}_{\alpha 3}\right]U^{(0)}_3\right\}$$

$$u_3 = u_3^{(0)} + \varepsilon^2 u_3^{(1)} + \varepsilon^4 u_3^{(2)}$$

$$= U_3^{(0)} + \varepsilon^2\left[b_{3\omega}^{(1)} U_\omega^{(0)} + b_{34}^{(1)} D_3^{(0)}(0) + b_{33}^{(2)} U_3^{(0)} + b_{34}^{(2)} \Phi^{(0)}\right]$$

$$+ \varepsilon^4\left\{\left[b_{3\omega}^{(3)} - b_{3\alpha}^{(2)} \bar{b}_{\alpha\omega}^{(1)}\right] U_\omega^{(0)} + \left[b_{33}^{(4)} - b_{3\alpha}^{(2)} \bar{b}_{\alpha 3}^{(2)}\right] U_3^{(0)}\right.$$

$$\left. + b_{33}^{(1)} \tau_{33}^{(1)}(0) + b_{34}^{(1)} D_3^{(1)}(0) + b_{34}^{(2)} \Phi^{(1)}\right\}$$

$$\varphi = \varphi^{(0)} + \varepsilon^2 \varphi^{(1)} + \varepsilon^4 \varphi^{(2)}$$

$$= \Phi^{(0)} + \varepsilon^2\left[b_{4\omega}^{(1)} U_\omega^{(0)} + b_{44}^{(1)} D_3^{(0)}(0) + b_{43}^{(2)} U_3^{(0)} + b_{44}^{(2)} \Phi^{(0)} + \Phi^{(1)}\right]$$

$$+ \varepsilon^4\left\{\left[b_{4\omega}^{(3)} - b_{4\alpha}^{(2)} \bar{b}_{\alpha\omega}^{(1)}\right] U_\omega^{(0)} + \left[b_{43}^{(4)} - b_{4\alpha}^{(2)} \bar{b}_{\alpha 3}^{(2)}\right] U_3^{(0)}\right.$$

$$\left. + b_{43}^{(1)} \tau_{33}^{(1)}(0) + b_{44}^{(1)} D_3^{(1)}(0) + b_{44}^{(2)} \Phi^{(1)}\right\} \quad (45)$$

More explicitly, using Eqs. (36), (38), (40), and (42), the closed-form analytical solution in the interior of the plate is

$$u_\omega = -\varepsilon\left[(q_3^+ - q_3^-) R_1^\omega x_1(x_1^2 - a^2) + l R_2^\omega x_1\right]$$

$$+ \varepsilon^3\left[(q_3^+ - q_3^-) R_7^\omega - m R_2^\omega\right] x_1$$

$$u_3 = k_3(x_1^2 - a^2)^2 + \varepsilon^2\left[D R_0^{12} + (q_3^+ - q_3^-) R_5^1(x_1^2 - \tfrac{1}{3}a^2) + l R_6^1\right]$$

$$+ \varepsilon^4\left[s_3 R_0^{11} + D' R_0^{12} - (q_3^+ - q_3^-) R_9^1 + m R_6^1\right] \quad (46)$$

for the displacement field,

$$\tau_{\alpha 3} = \varepsilon^2 (q_3^+ - q_3^-) R_4^\alpha x_1, \qquad \tau_{33} = \varepsilon^3\left[s_3 - (q_3^+ - q_3^-) R_8^1\right] \quad (47)$$

for the out-of-plane stress field, and

$$D_3 = \varepsilon(D + l R_3) + \varepsilon^3\left[D' - (q_3^+ - q_3^-) R_8^2 + m R_3\right]$$

$$\varphi = l(x_1^2 - a^2) + \varepsilon^2\left[D R_0^{22} + (q_3^+ - q_3^-) R_5^2(x_1^2 - \tfrac{1}{3}a^2) + l R_6^2\right.$$

$$\left. + m(x_1^2 - a^2)\right] + \varepsilon^4\left[s_3 R_0^{21} + D' R_0^{22} - (q_3^+ - q_3^-) R_9^2 + m R_6^2\right]$$

$$(48)$$

for the transverse electric displacement and the electric potential. Here

$$R_0^{\alpha\beta} = Q N^{\alpha\beta}, \qquad R_1^\omega = \tfrac{1}{6}(\tilde{d}_{\omega 3} - z\delta_{\omega 1}\tilde{d}_{33}), \qquad R_2^\omega = 2Q J_1^{\omega 2}$$

$$R_3 = 2Q K_{11}^{22}, \qquad R_4^\alpha = 6(Q - \bar{Q}) R_1^\omega L_{11}^{\alpha\omega}, \qquad R_5^\nu = 3Q R_1^\omega M_1^{\omega\nu}$$

$$R_6^\nu = Q(M_1^{\alpha\nu} R_2^\alpha + N^{\nu 2} R_3), \qquad R_7^\omega = Q I^{\omega\nu} R_4^\nu - 2Q J_1^{\omega\nu} R_5^\nu$$

$$R_8^\nu = Q J_1^{\omega\nu} R_4^\omega - \delta_{\nu 2} 2Q K_{11}^{22} R_5^2, \qquad R_9^\nu = Q(M_1^{\omega\nu} R_7^\omega + N^{\nu\lambda} R_8^\lambda)$$

$$(49)$$

The in-plane stress field and the in-plane electric displacement components, which may be discontinuous across interfaces, are determined from Eq. (11) to be

$$\tau_{\alpha\lambda} = \varepsilon\left[D M_\lambda^{\alpha 2} + l(M_\lambda^{\alpha 2} R_3 - L_{\lambda 1}^{\alpha\omega} R_2^\omega)\right.$$

$$\left. - (q_3^+ - q_3^-) L_{\lambda 1}^{\alpha\omega} R_1^\omega (3x_1^2 - a^2)\right] + \varepsilon^3\left[s_3 M_\lambda^{\alpha 1} + D' M_\lambda^{\alpha 2}\right.$$

$$\left. - (q_3^+ - q_3^-)(M_\lambda^{\alpha\nu} R_8^\nu - L_{\lambda 1}^{\alpha\omega} R_7^\omega) + m(M_\lambda^{\alpha 2} R_3 - L_{\lambda 1}^{\alpha\omega} R_2^\omega)\right]$$

$$D_\rho = -2l K_{1\rho}^{22} x_1 + \varepsilon^2\left[(q_3^+ - q_3^-)(J_\rho^{\alpha 2} R_4^\alpha - 2K_{1\rho}^{22} R_5^2) - 2m K_{1\rho}^{22}\right] x_1$$

$$(50)$$

Thus the exact closed-form solution of the problem has been obtained.

By setting the piezoelectric moduli $e_{kij} = 0$, we obtain results for a purely elastic strip that are slightly different from the solution of Ref. 21 because of the different choice of the reference planes. For a piezoelectric strip symmetric about the midplane, it can be verified that the present solution is exactly the same as that given in Ref. 12, where a symmetric elliptic plate is degenerated into an infinite strip.

The transverse shear stresses $\tau_{\alpha 3}$ given by Eq. (47) are independent of the relevant transverse shear moduli $c_{\alpha 3\omega 3}$, a property already known for elliptic monoclinic elastic plates.[20] Here this property is shown to be valid for monoclinic piezoelectric strips. In addition, the out-of-plane stresses $\tau_{\alpha 3}$ and τ_{33} in Eq. (47) do not depend on the normal electric displacement loading $D_3^\pm$ prescribed on the top and bottom surfaces of the plate. This means that the specified uniform electric displacements do not produce any out-of-plane stresses, another property particularly valid for the piezoelectric strips.

Differentiation of φ in Eq. $(48)_2$ with respect to x_1 and x_3 yields the following order of magnitude estimates of the electric field components:

$$E_1 = -\varphi_{,1} \sim (D_3^+ - D_3^-)\mathcal{O}(1) + (q_3^+ - q_3^-)\mathcal{O}(\varepsilon^2)$$

$$E_3 = -\varphi_{,3} \sim (D_3^+ - D_3^-)\mathcal{O}(\varepsilon) + (D_3^+ + D_3^-)\mathcal{O}(\varepsilon)$$

$$+ (q_3^+ - q_3^-)\mathcal{O}(\varepsilon) + (q_3^+ + q_3^-)\mathcal{O}(\varepsilon^3) \quad (51)$$

Thus, if $D_3^+ - D_3^- = 0$, the ratio of the in-plane to the out-of-plane electric field components is of the order of the dimensionless thickness parameter, i.e., $E_1/E_3 \sim \varepsilon$, which implies that for a thin piezoelectric plate E_1 is negligibly small as compared to E_3. On the other hand, E_1 should not be neglected when $D_3^+ - D_3^- \neq 0$ because it is of the order of the reciprocal of the thickness parameter, i.e., $E_1/E_3 \sim 1/\varepsilon$. In such a case the in-plane electric field component is in fact much larger than the out-of-plane component. Therefore the assumption that the in-plane electric field component is negligible is incorrect when the transverse electric displacements on the top and bottom surfaces are unequal to each other. However, for equal transverse electric displacements prescribed on the top and bottom surfaces, the existing thin piezoelectric plate models based on negligible in-plane electric field components are satisfactory in this aspect. This observation from the solution of laminated piezoelectric strips of arbitrary thickness agrees with that derived from the limit analysis[11] of thin single-layer plates.

For a piecewise homogeneous piezoelectric strip all of the material moduli are piecewise constants in the thickness direction. Observing Eqs. (46–50), the in-plane and out-of-plane distributions of the expansion terms of the mechanical displacements, stresses, electric displacements, and electric potential are summarized in Table 1. Of particular interest are the through-thickness distributions of the physical quantities. The through-thickness distributions of the expansion terms of the mechanical and electric quantities in z and ε are of the same form. Table 1 provides useful information for making proper approximations in two-dimensional piezoelectric plate models. In particular, the assumption of the through-thickness cubic distribution of the in-plane mechanical displacements in higher-order plate theories agrees with the present results. However, the constant distribution of the out-of-plane mechanical displacement provides only the zeroth-order approximation to our solution.

Table 1 In-plane and out-of-plane distributions of expansion terms of physical quantities of laminates

ε, z \ x_1	Constant	Linear	Quadratic	Cubic	Quartic
Constant	——	$D_\rho^{(0)}$	$\varphi^{(0)}$	$u_\omega^{(0)}$	$u_3^{(0)}$
Linear	$D_3^{(0)}$	——	$\tau_{\alpha\lambda}^{(0)}$	——	——
Quadratic	——	$\tau_{\alpha 3}^{(1)}, D_\rho^{(1)}$	$u_3^{(1)}, \varphi^{(1)}$	——	——
Cubic	$\tau_{33}^{(1)}, D_3^{(1)}, \tau_{\alpha\lambda}^{(1)}$	$u_\omega^{(1)}$	——	——	——
Quartic	$u_3^{(2)}, \varphi^{(2)}$	——	——	——	——

V. Numerical Results

The materials used to compute numerical results in the following examples are lead zirconate titanate[31] (PZT-4) and polyvinylidene fluoride[32] (PVDF). The material moduli of PZT-4 and PVDF are given in Table 2, where ε_0 is the permittivity of vacuum. Because a linear theory is used, results for complex loadings can be obtained by a superposition of the results for simple loadings. Two

Table 2 Values of nonvanishing material moduli for PZT-4 and PVDF

Moduli	PZT-4	PVDF
c_{1111}, GPa	139	238.24
c_{2222}, GPa	139	23.6
c_{3333}, GPa	115	10.64
c_{1122}, GPa	77.8	3.98
c_{1133}, GPa	74.3	2.19
c_{2233}, GPa	74.3	1.92
c_{2323}, GPa	25.6	2.15
c_{3131}, GPa	25.6	4.4
c_{1212}, GPa	30.6	6.43
e_{311}, C/m^2	−5.2	−0.13
e_{322}, C/m^2	−5.2	−0.145
e_{333}, C/m^2	15.1	−0.276
e_{223}, C/m^2	12.7	−0.009
e_{113}, C/m^2	12.7	−0.135
$\varepsilon_{11}/\varepsilon_0$[a]	1475	12.5
$\varepsilon_{22}/\varepsilon_0$[a]	1475	11.98
$\varepsilon_{33}/\varepsilon_0$[a]	1300	11.98

[a] $\varepsilon_0 = 8.854185$ pF/m.

Table 3 Results at some points for a clamped-clamped four-ply (PZT-4/90-deg PVDF/PZT-4/0-deg PVDF) laminate ($2a/h = 10$)

Parameter variation	Mechanical load q_3^+	Electric load D_3^+
$\bar{u}_1(a/2, h/2)$	0.02485	−0.005980
$\bar{u}_3(0, h/2)$	0.4595	−0.01078
$\bar{D}_1(a/2, -h/2)$	0.6177	−4.972
$\bar{D}_3(0, 0)$	−0.01637	0.4999
$\bar{\tau}_{11}[0, (h/4)^-]$	4.319	−0.2695
$\bar{\tau}_{22}[0, (h/4)^-]$	2.099	0.5102
$\bar{\varphi}(0, 0)$	0.07356	−0.2565

loading conditions are examined. One corresponds to applied uniform normal tractions with vanishing normal electric displacements and the other to applied uniform normal electric displacements with vanishing normal pressures. Results are presented in terms of the nondimensional variables

$$\bar{u}_i = u_i / Pa, \qquad \bar{\tau}_{ij} = \tau_{ij} / Pc^*$$
$$\bar{\varphi} = e^* \varphi / Pac^*, \qquad \bar{D}_i = D_i / Pe^* \tag{52}$$

with $c^* = 10^9$ N/m^2, $e^* = 1$ C/m^2, and either $P = q_3/c^*$ for applied normal traction q_3 or $P = D_3/e^*$ for applied normal electric displacement D_3.

A four-ply (PZT-4/90-deg PVDF/PZT-4/0-deg PVDF from bottom to top) piezoelectric laminate with equal thickness of each layer is examined. Selected results at some particular points of the laminated plate are listed in Table 3. The through-thickness distributions of the transverse shear stress $\bar{\tau}_{13}$ and the transverse normal stress $\bar{\tau}_{33}$ under applied normal traction q_3^+ are plotted in Figs. 1 and 2.

To illustrate the order of magnitude of the electric field components, we consider a homogeneous single-layer PZT-4 plate with normal electric displacements $D_3^{\pm}$ prescribed on its top and bottom surfaces. For symmetric electric loading $D_3^+ = D_3^-$ the ratio of the in-plane to the out-of-plane electric field components E_1/E_3 can be shown to be identically zero. For antisymmetric electric loading $D_3^+ = -D_3^-$ we have plotted in Fig. 3 the through-thickness distribution of $E_1/E_3 = \beta x_1/a$ within the lower half plate, as the result for the upper half plate is antisymmetric about the midplane. Here

$$\beta = 2a / \varepsilon (M_1^{\alpha 2} R_2^{\alpha} + N^{22} R_3) \tag{53}$$

is a function of x_3. Because $E_3 = 0$ at the midplane of the plate, the value of β approaches infinity as $x_3 \to 0^-$ and thus is not shown therein for $-0.25 < 2x_3/h < 0$. Figure 3 shows that for this case the in-plane electric field component cannot be neglected as compared with the out-of-plane electric field component.

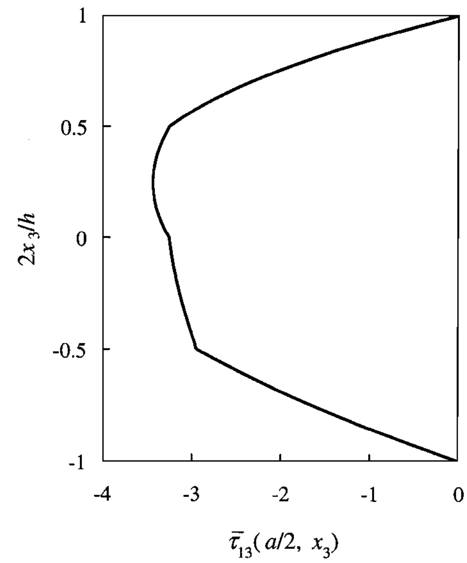

Fig. 1 Through-the-thickness distribution of the dimensionless transverse shear stress for a four-ply (PZT-4/90-deg PVDF/PZT-4/0-deg PVDF) laminate under q_3^+ ($2a/h = 10$).

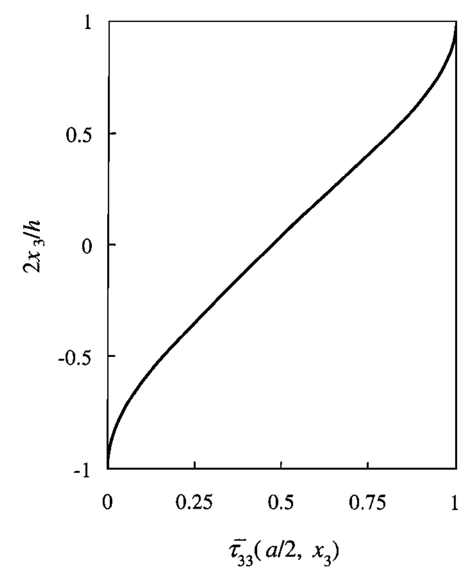

Fig. 2 Through-the-thickness distribution of the dimensionless transverse normal stress for a four-ply (PZT-4/90-deg PVDF/PZT-4/0-deg PVDF) laminate under q_3^+ ($2a/h = 10$).

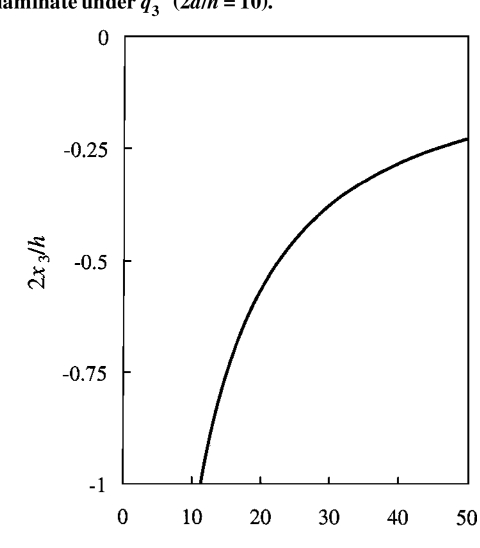

Fig. 3 Through-the-thickness distribution of the ratio of the in-plane to the out-of-plane electric field components for a homogeneous PZT-4 plate under $D_3^+ = -D_3^-$ ($2a/h = 10$).

Although the numerical results for only the span-to-thickness ratio $2a/h = 10$ are given, many other results can simply be obtained in terms of expressions of physical quantities and Table 1. For example, for $2a/h = 100$ the value of β will be 10 times the value given in Fig. 3 for $2a/h = 10$ according to Eq. (53), and τ_{33} will be the same at a plane parallel to the midplane according to Table 1.

VI. Conclusions

We have combined the method of asymptotic series expansion with the transfer matrix method to obtain a closed-form solution for the three-dimensional electromechanical deformations of a clamped-clamped laminated plate. The governing equations are expressed in terms of functions defined on the midplane of the plate. Equations for the determination of the nth-order mechanical displacements and electric potentials involve effective loads that depend upon the quantities of order 0 through $(n-1)$. For a clamped-clamped piezoelectric strip of length $2a$, these are further simplified, and a closed-form solution of the governing equations is derived. The transverse shear stresses $\tau_{\alpha 3}$ $(\alpha = 1, 2)$ are independent of the relevant transverse shear moduli $c_{\alpha 3 \omega 3}$, and $\tau_{\alpha 3}$ and the transverse normal stress τ_{33} do not depend upon the uniform normal electric displacement $D_3^{\pm}$ prescribed on the top and bottom surfaces of the laminate. For $D_3^+ = D_3^-$ the ratio of the in-plane to the out-of-plane electric field components is of the order of $\varepsilon = h/2a$, where h is the thickness of the laminate. However, if $D_3^+ - D_3^- \neq 0$, then the in-plane electric field is much higher than the out-of-plane electric field and cannot be neglected as is often done in some plate theories. Guidelines for making proper approximations for the mechanical and electric quantities in two-dimensional plate theories are provided in Table 1. In particular, the assumption of the through-thickness cubic variation of the in-plane mechanical displacements agrees with the closed-form solution derived here for the cylindrical bending deformations of a piezoelectric laminate.

Numerical results are presented for a piezoelectric laminated strip subjected to either mechanical or electric loads on the top and bottom surfaces.

The present solution could serve for checking the validity of other approximate two-dimensional theories and numerical methods.

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